UDC 539.3

CONTACT PROBLEM FOR A CIRCULAR PLATE ON AN ELASTIC FOUNDATION

PMM Vol. 43, No. 2, 1979, pp. 330-334
S. N. KARASEV
(Kazan')
(Received May 26, 1978)

The problem of bending of a circular plate on an elastic foundation a system of rigid annular stamps with a flat bottom is considered. On the basis of the Kirchhoff — Love theory, integral equations are constructed for the desired normal contact stresses with the transverse compression of the plate in thickness taken into account in the contact zones [1 —3]. The integral equations are reduced to systems of linear algebraic equations which are separated into groups of independent equations.

Primarily two approaches are used to solve the contact problems of the theory of thin plates and shells. The first is to construct solutions of the differential equations of the theory of plates and shells in the contact domains and outside them, and to join these solutions on the boundaries of the contact zones. The solution of a number of contact problems for plates and shells has been obtained on the basis of this approach [4 — 13]. The second approach is based on the construction of integral equations in the desired contact stresses and in determining their solutions [1, 14] (*).

If the contact zone is a domain of complex shape (trapezoid, triangle, ellipse, etc.), then in practice it is difficult to satisfy the conditions for joining the solutions on the boundary of this zone. Problems in which the contact zone boundaries are unknown afford the greatest difficulty; let us just note the paper [15] in which a solution is given for the two-dimensional contact problem for a circular plate.

A method of solving contact problems, formulated in the form of integral equations, is proposed in an example of solving a contact problem for a circular plate on an elastic foundation and subjected to a system of m annular stamps with flat bottoms (the contact domain is known in advance). The kernels of these integral equations are the fundamental solutions of the differential equations of plate and shell theory. Normal contact stresses are determined under the assumption that there are no tangential contact stresses and no zone of separation of the plate from the stamp.

Let us consider the bending of a circular plate on an elastic Winkler foundation. Let us take a polar r, θ coordinate system in the middle plane and its origin at the center of a circle on which the side surface of the plate intersects the middle plane. In this case the plate bending equation has the form

*) See also Tolkachev, V. M., Some Contact Problems of Shell Theory. Doctoral dissertation. Moscow, 1973.

$$\Delta\Delta w + w = \frac{\omega^2}{Dx}\delta(x-x_0)\delta(\theta-\theta_0), \quad \omega^4 = \frac{D}{K}, \quad x = \frac{r}{\omega} \tag{1}$$
$$D = \frac{Eh^3}{12(1-\nu^2)}, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} + \frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}$$

Here K is the elastic modulus of the foundation, E, ν are the elastic modulus and Poisson's ratio of the plate, h is the plate thickness, and δ is the delta function.

Using the results in [16], let us write the general solution of (1) in the form

$$w = \sum_{n=0}^{\infty}(S_n{}^0\cos n\theta + S_n{}^1\sin n\theta) + G(x, \theta, x_0, \theta_0)$$
$$S_n{}^k = C_n{}^k u_n + B_n{}^k v_n + D_n{}^k f_n + A_n{}^k g_n, \quad k = 0, 1$$
$$G(x, \theta, x_0, \theta_0) = \sum_{n=0}^{\infty} b_n K_n(x, x_0)\cos n(\theta-\theta_0)$$
$$n = 0, \quad b_n = {}^1/_2; \quad n \geqslant 1, \quad b_n = 1$$
$$K_n(x, x_0) = \omega^2 H(x-x_0)\, Q_n(x, x_0)\,/\,(2D)$$
$$Q_n(x, x_0) = f_n(x)\,u_n(x_0) - f_n(x_0)\,u_n(x) - g_n(x)\,v_n(x_0) + g_n(x_0)\,v_n(x)$$

Here $H(x-x_0)$ is the unit function, u_n, v_n, f_n, g_n are Kelvin functions related to Bessel functions J_n of the first and $H_n{}^{(1)}$ of the third kinds by the formulas [16]

$$u_n + iv_n = J_n(x\sqrt{i}), \quad f_n + ig_n = H_n{}^{(1)}(x\sqrt{i})$$

Considering a load in the form of the lumped force P to be applied to the plate at a point with coordinates (x_1, θ_1) and taking account of the transverse strain compression of the plate in thickness in the contact zones [3], we write the integral equations for the unknown contact stresses $\sigma_j(x, \theta)$ as

$$\alpha\sigma_j(x, \theta) + \omega^2 \sum_{i=1}^{m}\int_0^{2\pi}\int_{a_i}^{b_i} G(x, \theta, x_0, \theta_0)\,\sigma_i(x_0, \theta_0)\,x_0\,dx_0\,d\theta_0 = U_j \tag{2}$$
$$U_j = \gamma_j + \beta_{1j}\,x\cos\theta + \beta_{2j}x\sin\theta - PG(x, \theta, x_1, \theta_1) - V_j$$
$$V_j = \sum_{n=0}^{\infty}(S_n{}^0\cos n\theta + S_n{}^1\sin n\theta)$$
$$a_j < x < b_j, \quad 0 \leqslant \theta \leqslant 2\pi; \quad \alpha = 13(1-\nu^2)\,h\,/\,(32E);$$
$$j = 1, \ldots, m$$

Here the constants γ_j, β_{1j}, β_{2j} characterize the displacement of the j-th stamp as a rigid body, and a_j, b_j are the dimensionless internal and external radii of the j-th stamp.

The stamp equilibrium conditions

$$\omega^3\int_0^{2\pi}\int_{a_i}^{b_i}\sigma_i x^2\cos\theta\,dx\,d\theta = M_{1i} \quad (i = 1, \ldots, m) \tag{3}$$

$$\omega^3 \int_0^{2\pi} \int_{a_i}^{b_i} \sigma_i x^2 \sin\theta \, dx \, d\theta = M_{2i}, \quad \omega^2 \int_0^{2\pi} \int_{a_i}^{b_i} \sigma_i x \, dx \, d\theta = P_i$$

are appended to (2). Here P_i, M_{1i}, M_{2i} are projections of the principal vectors and moments of the external forces applied to the stamps, on the coordinate axes.

Acting on (2) with the operator $(\Delta\Delta + 1)$ and using the filtering property of the delta-function, we find that the functions σ_j satisfy the equations

$$\Delta\Delta\sigma_j + (\lambda^4 + 1)\,\sigma_j = \alpha^{-1}(\gamma_j + \beta_{1j}\, x \cos\theta + \beta_{2j} x \sin\theta) \tag{4}$$
$$\lambda^4 = \omega^4 / (\alpha D)$$

Solving (4), we find

$$\sigma_j(x, \theta) = \alpha^{-1}\varepsilon^{-4}(\gamma_j + \beta_{1j} x \cos\theta + \beta_{2j} x \sin\theta) + \tag{5}$$
$$\sum_{n=0}^{\infty} (\overset{\circ}{\sigma}_{ni}(x) \cos n\theta + \sigma_{ni}^1(x) \sin n\theta), \quad \varepsilon^4 = 1 + \lambda^4$$

The functions $\sigma_{ni}{}^\circ$ and $\sigma_{ni}{}^1$ are determined from the equations

$$\Delta_n\Delta_n\sigma_{ni} + \varepsilon^4\sigma_{ni} = 0, \quad \Delta_n = \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{n^2}{x^2} \tag{6}$$
$$\sigma_{ni}{}^\circ(x) = A_{ni}u_n(\varepsilon x) + B_{ni}v_n(\varepsilon x) + C_{ni}f_n(\varepsilon x) + D_{ni}g_n(\varepsilon x)$$
$$\sigma_{ni}{}^1(x) = A_{ni}{}^1 u_n(\varepsilon x) + B_{ni}{}^1 v_n(\varepsilon x) + C^1{}_{ni} f_n(\varepsilon x) + D^1{}_{ni} g_n(\varepsilon x)$$

The differential equations (4) govern the structure of the solutions of the integral equations (2). Taking into account that the functions σ_j satisfy (4), we reduce the integral equations (2) to a system of linear algebraic equations.

We obtain from (4) — (6)

$$\sigma_i(x, \theta) = -\lambda^{-4}\sigma_i(x, \theta) - \lambda^{-4} \sum_{n=0}^{\infty} (\Delta_n\Delta_n \overset{\circ}{\sigma}_{ni} \cos n\theta + \tag{7}$$
$$\Delta_n\Delta_n\sigma_{ni}^1 \sin n\theta) + \lambda^{-4}\alpha^{-1}(\gamma_i + \beta_{1i} x \cos\theta + \beta_{2i} x \sin\theta)$$

Substituting (7) into (2) and integrating by parts, we arrive at the following expression

$$\alpha\sigma_j + \pi\omega^2\lambda^{-4} \sum_{n=0}^{\infty} \sum_{i=1}^{m} b_n x_0 [\overset{\circ}{\Lambda}_{ni} \cos n\theta + \Lambda_{ni}^1 \sin n\theta] + \tag{8}$$
$$\sum_{i=1}^{m} [\pi\omega^2\lambda^{-4}\alpha^{-1} x_0 \Omega_i - \omega^4\lambda^{-4}\Pi_i] = -P \sum_{n=0}^{\infty} \beta_n - V_j$$
$$\Lambda_{ni}^k = \left[\sigma_{ni}^k L_n K_n - K_n L_n \sigma_{ni}^k + \frac{dK_n}{dx_0} L\sigma_{ni}^k - \frac{d\sigma_{ni}^k}{dx_0} LK_n\right]\Bigg|_{a_i}^{b_i}, \quad k = 0, 1$$
$$\Omega_i = \left[\left(LK_1 - x_0\frac{dK_1}{dx_0} - x_0 L_1 K_1 - \frac{K_1}{x_0}\right)(\beta_{1i}\cos\theta + \beta_{2i}\sin\theta) - \right.$$
$$\left. \frac{1}{2}\gamma_i L_0 K_0\right]\Bigg|_{a_i}^{b_i}$$

$$\Pi_i = \int_0^{2\pi}\int_{a_i}^{b_i} [(\Delta\Delta + 1)G]\, x_0\sigma_i\, dx_0\, d\theta_0$$
$$\beta_n = b_n K_n(x,\ x_1)\cos n(\theta - \theta_1)$$
$$L = \frac{d^2}{dx_0^2} + \frac{1}{x_0}\frac{d}{dx_0}, \quad L_n y = \frac{d}{dx_0} Ly - n^2 \frac{d}{dx_0}\left(\frac{y}{x_0^2}\right) - \frac{n^2}{x_0}\frac{dy}{dx_0}$$
$$n = 0, 1, 2, \ldots$$

Only four systems of linearly independent functions are contained in (8):

$$\begin{bmatrix}\cos n\theta\\ \sin n\theta\end{bmatrix} u_n(x), \quad \begin{bmatrix}\cos n\theta\\ \sin n\theta\end{bmatrix} v_n(x), \quad \begin{bmatrix}\cos n\theta\\ \sin n\theta\end{bmatrix} f_n(x), \quad \begin{bmatrix}\cos n\theta\\ \sin n\theta\end{bmatrix} g_n(x)$$

Equating the coefficients of identical functions on the left and right in (8), and taking account of (6), we obtain the first infinite system of linear algebraic equations in the coefficients $A_{ni}, \ldots, D_n^k$ in the form

$$\sum_{i=1}^{m}\left[b_n\Lambda_{ni}^-(b_i,\ a_i) + \alpha^{-1}d_n\beta_{1i}\Phi_{ni}^-(b_i,\ a_i) - \frac{1}{2}\gamma_i\omega_n L_0^-(b_i,\ a_i)\right] + \tag{9}$$
$$D_n^k = \frac{1}{2D} P b_n \omega^2 H(x - x_1)\, u_n(x_1)\cos n\theta_1, \quad k = 0$$
$$\Lambda_{ni}^-(b_i,\ a_i) = b_i\,\Lambda_{ni}^*(b_i)\, H(x - b_i) - a_i\Lambda_{ni}^*(a_i)\, H(x - a_i)$$
$$\Lambda_{ni}^*(z) = \sigma_{ni}(z)\, L_n u_n(z) - u_n(z)\, L_n\sigma_{ni}(z) + L\sigma_{ni}(z)\frac{du_n(z)}{dz} - \frac{d\sigma_n(z)}{dz} Lu_n(z)$$
$$\Phi_{ni}^-(b_i,\ a_i) = \Phi_{ni}^*(b_i) - \Phi_{ni}^*(a_i)$$
$$\Phi_{ni}^*(z) = zLu_1(z) - z^2\frac{du_1(z)}{dz} - z^2 L_1 u_1(z) - u_1(z), \quad z = a_i,\ b_i$$
$$L_0^-(b_i,\ a_i) = b_i L_0 u_0(b_i) - a_i L_0 u_0(a_i)$$
$$n = 1, \quad d_n = 1; \quad n = 0,\ 2,\ 3,\ \ldots,\quad d_n = 0$$
$$n = 0, \quad \omega_n = 1; \quad n = 1,\ 2,\ 3,\ \ldots,\quad \omega_n = 0$$

We obtain the second, third, and fourth system of equations from (9) by sequential replacement of $u_n(z)$ by $g_n(z)$, $-f_n(z)$, $-v_n(z)$ $(z = a_i,\ b_i)$ and D_n^k by B_n^k, C_n^k, A_n^k.

The equilibrium conditions reduce to the system of equations

$$\psi_i^-(b_i,\ a_i) + \beta_{ki}(b_i^3 - a_i^3)/(3\alpha) = -M_{ki}\varepsilon^4/(\pi\omega^3), \quad k = 1, 2 \tag{10}$$
$$a_i L_0\sigma_{0i}(a_i) - b_i L_0\sigma_{0i}(b_i) + \gamma_i(b_i^2 - a_i^2)/(2\alpha) = P_i\varepsilon^4/(2\pi\omega^2)$$
$$\psi_i^-(b_i,\ a_i) = \psi_i^*(b_i) - \psi_i^*(a_i)$$
$$\psi_i^*(z) = z^2 L_1\sigma_{1i}^k(z) + \frac{d}{dx}\sigma_{1i}^k(z) + \sigma_{1i}^k(z) - zL\sigma_{1i}^k(z), \quad k = 0, 1$$

We append boundary conditions to (9) and (10). We assume that the edges of the annular plates are clamped, i.e., $w(R_i, \theta) = 0, \partial w(R_i, \theta) / \partial x = 0$ ($i = 1$, 2; R_1, R_2 are the inner and outer dimensionless radii of the plate). We obtain from the boundary conditions $w(R_i, \theta) = 0$

$$\sum_{i=1}^{m} \Big\{ b_n [\Lambda_{ni}^{**}(b_i) - \Lambda_{ni}^{**}(a_i)] + \alpha^{-1} d_n \beta_{1i} [\Phi_{ni}^{**}(b_i) - \Phi_{ni}^{**}(a_i)] - \tag{11}$$

$$\frac{1}{2} \gamma_i \omega_n L_0^{**}(b_i, a_i) \Big\} + S_n^{0}(R_j) + P b_n K_n(R_j, x_1) \cos n\theta_1 = 0$$

$$n = 0, 1, 2, \ldots; \quad j = 1, 2$$

Here the functions $\Lambda_{ni}^{**}(z)$, $\Phi_{ni}^{**}(z)$, $L_0^{**}(b_i, a_i)$ are obtained from $\Lambda_{ni}^{*}(z)$, $\Phi_{ni}^{*}(z)$, $L_0^{-}(b_i, a_i)$ by replacing $u_n(z)$ by $K_n(R_j, z)$. In order to obtain two other equations it is sufficient to replace $K_n(R_j, b_i)$ in (11) by $dK_n(R_j, b_i) / dx$ and $K_n(R_j, a_i)$ by $dK_n(R_j, a_i) / dx$ and the functions u_n, v_n, f_n, g_n by their derivatives. For arbitrary n the infinite system of equations (9)–(11) is divided into finite groups of equations. Each group of equations contains not more than $5m + 5$ equations.

The main advantage of the method considered for solving the contact problem as compared with the juncture method is the reduction in the number of arbitrary constants to be determined. The method mentioned does not require the construction of influence functions [1, 14].

Let the annular plate be loaded axisymmetrically by six stamps. The first approach results in this case in the need to solve a system of 58 linear algebraic equations. The method developed here reduces this problem to the solution of 34 equations. The advantage of the method mentioned becomes evident as the number of contact zones increases.

REFERENCES

1. Popov, G. Ia. On contact problems for shells and plates. Trudy, Tenth All-Union Conf. on Plate and Shell Theory, Vol. 1, Kutaisi, 1975. Metsniereba, Tbilisi, 1975.

2. Grigoliuk, E. I. and Tolkachev, V. M., Cylindrical bending of a plate by rigid stamps, PMM, Vol. 39, No. 5, 1975.

3. Karasev, S. N. and Ariukhin, Iu. P., Influence of transverse shear and compression on the contact stress distribution. IN: Investigations on the Theory of Plates and Shells, No. 12, Izd. Kazań. Univ. 1976.

4. Aleksandrov, V. M., Some contact problems for beams, plates, and shells. Inzh. Zh., Vol. 5, No. 4, 1965.

5. Artiukhin, Iu. P. and Karasev, S. N., Effect of a rigid stamp on a shallow spherical shell and plate, No. 9, Izd. Kazań. Univ. 1972.

6. Artiukhin, Iu. P. and Karasev, S. N., Stress determination in a shallow spherical shell under the effect of a rigid body. IN: Investigations on the Theory of Plates and Shells, No. 11. Izd. Kazań. Univ. 1975.

7. B l o k h, M. V. and T s u k r o v, S. Ia., On the influence of a wall thickness change on the axisymmetric contact of thin cylindrical shells. Prikl. Mekhan., Vol. 10, No. 4, 1974.

8. B o n d a r e n k o, V. A., Contact of a thin - walled spherical shell with a rigid ball, Prikl. Mekhan., Vol. 7, No. 12, 1971.

9. K r u k, G. S., Contact problem for a sandwich plate with a light core, IN: Mathematical Methods and Physico-mathematical Fields, No. 1, "Naukova Dumka", Kiev, 1975.

10. L u k a s h, P. A. and L e o n t ' e v, N. M., Analysis of a spherical shell resting on a rigid foundation Inzh. Sb., Vol. 25, 1959.

11. P e l e k h, B. L. and S u k h o r o l ' s k i i, M. A., Interaction of systems of rigid smooth stamps with elastic cylindrical shells of bonded plastics, Prikl. Mekhan., Vol. 10, No. 4, 1974.

12. R o z e n b e r g, L. A., On the pressure of a solid on a plate, Inzh. Sb., Vol. 21. 1955.

13. E s s e n b u r g, F., On surface constraints in plate problems. Trans. ASME, Ser. E., J. Appl. Mech., Vol. 29, No. 2, 1962.

14. P e l e k h, B. L. and S u k h o r o l ' s k i i, M. A., On the solution of elastic contact problems of cylindrical shells, Prikl. Mekhan. Vol. 10, No. 8, 1974.

15. G a l i n, L. A., On the pressure of a solid on a plate, PMM, Vol. 12, No. 2, 1948.

16. K o r e n e v. B. G., Introduction to the Theory of Bessel Functions. "Nauka", Moscow, 1971.

Translated by M. D. F.